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Some New Topological Features of Feynman Graphs

ALLADI RAMAKRISHNAN

MATSCIENCE, Institute of Mathematical Sciences, Madras, India

In this contribution we bring to focus some unnoticed topological features of Feynman graphs hitherto obscured by the extraordinary emphasis on manifest covariance at every stage of the description of a quantum mechanical process. It is well known that the equality of the perturbation expansions in the Feynman formalism to those in covariant field theory was achieved only after painstaking efforts through laborious and longwinded arguments. The aim in such attempts had been to establish the equality of a single term corresponding to a Feynman diagram with n vertices to the sum of $n!$ terms in field theory. Such equality is not so "manifest" as is the correspondence between the Feynman propagators and the field theoretical commutators.

Our object now is to show that if only we decompose the propagators into positive and negative energy "arms," a single Feynman diagram splits into 2^{n-1} diagrams which we call *patterns* and the correspondence between the 2^{n-1} *patterns* and $n!$ terms of field theory can be made "manifest" in a manner as to enhance the "topological beauty" of a Feynman diagram. *The fact that the terms corresponding to patterns are not covariant should not worry us any more than the noncovariance of $n!$ terms since the covariance is preserved for the sums in both the cases.*

This idea of decomposition of the propagator was suggested and used by this author and his collaborators earlier in a series of papers [1, 2, 3, 4], but we were deterred in the pursuit of our attempts when confronted by some puzzling features. We shall now show that considerable insight can be gained if we recognize that the four-dimensional transforms of singular functions occurring in perturbation theory can be obtained in *two stages*, a three-dimensional transformation over space, followed by a transformation over time. *This quite naturally leads to the decomposition of the propagator.* The use of a simple lemma in complex variable theory combined with the conservation law of energy resolves the "puzzle" and strikingly brings to light new facets of the topological structure of Feynman diagrams. The energy conservation is expressed as the vanishing of the sum of energy *imbalances* associated with vertices rather than with propagators, an idea introduced by this author 3 years ago [5].

It is hoped that such an insight into the topological features will facilitate the study of analytic properties of Feynman amplitudes and their physical interpretation. We are encouraged in this attempt by an elegant formulation of charge conjugation in Feynman formalism suggested as a natural consequence of this new approach.

Thus this paper will be divided into six parts:

1. Space and time transforms of singular functions.
2. Decomposition of the Fermion propagators.
3. Energy imbalance of a vertex.
4. Composition of imbalances.
5. Application to a process of the fifth order.
6. Charge conjugation in Feynman formalism.

1. Space and Time Transforms of Singular Functions

The Fourier transform $\phi(s)$ of a function $f(t)$ is defined as

$$\phi(s) = \int_{-\infty}^{+\infty} f(t) e^{\pm ist}\, dt, \tag{1}$$

the $\pm$ sign of the exponent being chosen according to convenience. But this choice will be reflected in the inverse transformation as follows:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(s) e^{\mp ist}\, dt. \tag{2}$$

The Fourier transforms for functions of more than one variable can be defined accordingly.

In (1) we shall use the negative sign in the exponent in defining the space transform and the positive sign when defining the transform with respect to time. Correspondingly we shall have to use the opposite sign in defining the inverse transform. This is because our scalar product of the four momentum p and space-time vector x is taken as

$$p \cdot x = p_4 t - \vec{p} \cdot \vec{x}.$$

Let us consider the invariant singular functions of field theory.

(i) We shall first take up the functions which satisfy the homogeneous Klein-Gordon equation.

(a)
$$\begin{aligned} \Delta(x) &= (-i) \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d^3p}{2E} [e^{-ip\cdot x} - e^{+ip\cdot x}] \\ &= (-i) \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d^3p}{2E} [e^{-iEt+i\vec{p}\cdot\vec{x}} - e^{iEt-i\vec{p}\cdot\vec{x}}] \\ &= (-i) \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d^3p}{2E} [e^{-iEt} - e^{+iEt}] e^{+i\vec{p}\cdot\vec{x}}, \end{aligned} \tag{3}$$

where $E = +\sqrt{\vec{p}^2 + m^2}$, m being the mass of the particle. If $D(\vec{p}, t)$ is the three-dimensional transform of $\Delta(x)$ is defined by

$$D(\vec{p}, t) = \int_{-\infty}^{+\infty} \Delta(x) e^{-i\vec{p}\cdot\vec{x}}\, d^3x \tag{4}$$

from a comparison of (2) and (3) we recognize that

$$D(\vec{p}, t) = (-i)\frac{1}{2E}[e^{-iEt} - e^{+iEt}]. \tag{5}$$

(b)

$$\Delta^+(x) = (-i)\frac{1}{(2\pi)^3}\int_{\substack{-\infty\\ E>0}}^{+\infty} \frac{d^3p}{2E} e^{-ip\cdot x} \tag{6}$$

$$\Delta^-(x) = (-i)\frac{1}{(2\pi)^3}\int_{\substack{-\infty\\ E>0}}^{+} \frac{d^3p}{(-2E)} e^{+ip\cdot x} \tag{7}$$

and

$$\Delta(x) = \Delta^+(x) + \Delta^-(x). \tag{8}$$

It is easy to recognize the space transforms, $D^+(\vec{p}, t)$ and $D^-(\vec{p}, t)$ of the functions $\Delta^+(x)$ and $\Delta^-(x)$. They are given by

$$D^+(\vec{p}, t) = (-i)\frac{1}{2E} e^{-iEt} \tag{9}$$

$$D^-(\vec{p}, t) = (-i)\frac{1}{(-2E)} e^{+iEt}. \tag{10}$$

(ii) Let us consider now the functions $\Delta_F(x)$, $\Delta_A(x)$, and $\Delta_R(x)$ which satisfy the inhomogeneous Klein-Gordon equation.

(a)

$$\begin{aligned} \Delta_F(x) &= 2i\,\Delta^+(x) \qquad \text{for} \quad t > 0 \\ &= -2i\,\Delta^-(x) \qquad \text{for} \quad t < 0. \end{aligned} \tag{11}$$

This may be written as

$$\Delta_F(x) = 2i[\theta(t)\,\Delta^+(x) - \theta(-t)\,\Delta^-(x)], \tag{12}$$

where $\theta(t)$ is the Heaviside unit function, viz.,

$$\begin{aligned} \theta(t) &= 1; \qquad t > 0 \\ &= 0; \qquad t < 0. \end{aligned} \tag{13}$$

The three-dimensional transform $D_F(\vec{p}, t)$ of $\Delta_F(x)$ is easily recognized as

$$D_F(\vec{p}, t) = (2i)(-i)\left[\frac{\theta(t)}{2E} e^{-iEt} - \frac{\theta(-t)}{(-2E)} e^{+iEt}\right]. \tag{14}$$

(b) $\Delta_R(x)$ is defined as

$$\begin{aligned} \Delta_R(x) &= -\Delta(x) \qquad \text{for} \quad t > 0 \\ &= 0 \qquad\qquad \text{for} \quad t < 0, \end{aligned} \tag{15}$$

and its three-dimensional transform $D_R(\vec{p}, t)$ is given by

$$D_R(\vec{p}, t) = -(-i)\frac{\theta(t)}{2E}[e^{-iEt} - e^{+iEt}]. \tag{16}$$

(c) $\Delta_A(x)$ is defined as

$$\begin{aligned} \Delta_A(x) &= 0 \qquad\qquad \text{for} \quad t > 0 \\ &= \Delta(x) \qquad \text{for} \quad t < 0, \end{aligned} \tag{17}$$

and $D_A(\vec{p}, t)$ is given by

$$D_A(\vec{p}, t) = (-i)\frac{\theta(-t)}{2E}[e^{-iEt} - e^{+iEt}]. \tag{18}$$

We now perform the transformation with respect to the fourth or time component. We shall multiply the three-dimensional transforms by e^{+ip_4t} and integral over t.

(a) We shall denote the temporal transform of $D(\vec{p}, t)$ by $D(p)$. Thus

$$\begin{aligned} D(p) &= \int_{-\infty}^{+\infty} D(\vec{p}, t)e^{+ip_4t}\,dt \\ &= (-i)\int_{-\infty}^{+\infty}\frac{1}{2E}[e^{-iEt} - e^{+iEt}]e^{+ip_4t}\,dt \\ &= (-i)\frac{2\pi}{2E}[\delta(p_4 - E) - \delta(p_4 + E)] \\ &= (-i)(2\pi)\delta(p_4^2 - E^2)\epsilon(p_4) \\ &= (-i)(2\pi)\delta(p^2 - m^2)\epsilon(p_4), \end{aligned} \tag{19}$$

where δ is the Dirac delta function and

$$\begin{aligned} \epsilon(p_4) &= 1 \qquad\quad \text{if} \quad p_4 > 0 \\ &= 0 - 1 \qquad \text{if} \quad p_4 < 0. \end{aligned} \tag{20}$$

(b)
$$\begin{aligned} D_F(p) &= \int_{-\infty}^{+\infty} D_F(\vec{p}, t)e^{+ip_4t}\,dt \\ &= (2i)(-i)\int_{-\infty}^{+\infty}\left[\frac{\theta(t)}{2E}e^{-iEt} - \frac{\theta(-t)}{(-2E)}e^{+iEt}\right]e^{+ip_4t}\,dt. \end{aligned} \tag{21}$$

To ensure convergence we shall multiply the first part of the integral by $e^{-\epsilon t}$ and the second by $e^{+\epsilon t}$ and take the limit as $\epsilon \to 0$. Thus

$$
\begin{aligned}
D_F(p) &= (2i)(-i)\int_{-\infty}^{+\infty}\left[\frac{\theta(t)}{2E}\,e^{-iEt}e^{-\epsilon t} - \frac{\theta(-t)}{(-2E)}\,e^{+iEt}e^{+\epsilon t}\right]e^{+ip_4 t}\,dt \\
&= (2i)(-i)\left\{\left[\frac{1}{2E}\,\frac{e^{i(p_4-E+i\epsilon)t}}{i(p_4-E+i\epsilon)}\right]_0^{\infty} - \left[\frac{1}{(-2E)}\,\frac{e^{i(p_4+E-i\epsilon)}}{i(p_4+E-i\epsilon)}\right]_{-\infty}^{0}\right\} \\
&= \frac{(2i)(-i)}{2E}\left[\frac{-1}{i(p_4-E+i\epsilon)} + \frac{1}{i(p_4+E-i\epsilon)}\right] \\
&= 2i\,\frac{1}{p_4^2-(E-i\epsilon)^2}\,.
\end{aligned}
\tag{22}
$$

(c) Let us denote by $D_R(p)$ the time transform of $D_R(\vec{p}, t)$

$$
\begin{aligned}
D_R(p) &= \int_{-\infty}^{+\infty} D_R(\vec{p}, t)e^{+ip_4 t}\,dt \\
&= -(-i)\int_{-\infty}^{+\infty}\frac{\theta(t)}{2E}\,[e^{-iEt} - e^{+iEt}]e^{+ip_4 t}\,dt \qquad (23) \\
&= (-i)\int_0^{\infty}\frac{1}{2E}\,[e^{-iEt} - e^{+iEt}]e^{+ip_4 t}\,dt.
\end{aligned}
$$

The integral converges if we introduce a factor $e^{-\epsilon t}$ and redefine $D_R(p)$ as follows:

$$
D_R(p) = -(-i)\int_0^{\infty}\frac{1}{2E}\,[e^{-iEt} - e^{+iEt}]e^{+ip_4 t}e^{-\epsilon t}\,dt \tag{24}
$$

and take the limit as $\epsilon \to 0$.
Thus

$$
\begin{aligned}
D_R(p) &= (-1)(-i)\,\frac{1}{2E}\left[\frac{e^{i(p_4-E+i\epsilon)t}}{i(p_4-E+i\epsilon)} - \frac{e^{i(p_4+E+i\epsilon)}}{i(p_4+E+i\epsilon)}\right]_0^{\infty} \\
&= (-1)(-i)(-1)\,\frac{1}{2E}\left[\frac{1}{i(p_4-E+i\epsilon)} - \frac{1}{i(p_4+E+i\epsilon)}\right] \qquad (25) \\
&= -\,\frac{1}{(p_4+i\epsilon)^2-E^2}\,.
\end{aligned}
$$

(d) In a similar way $D_A(p)$ is obtained. Here the factor we use to make integral convergent is $e^{+\epsilon t}$ as $\epsilon \to 0$

$$
\begin{aligned}
D_A(p) &= (-i)\int_{-\infty}^{0}\frac{1}{2E}\,[e^{-iEt} - e^{+iEt}]e^{ip_4 t}e^{+\epsilon t}\,dt \qquad \text{as} \qquad \epsilon \to 0 \\
&= +\,\frac{1}{(p_4-i\epsilon)^2-E^2}\,.
\end{aligned}
\tag{26}
$$

2. Decomposition of the Fermion Propagator[1]

The Fermion propagator is defined by

$$K_F(x) = \tfrac{1}{2}(i\nabla + m)\,\Delta_F(x), \tag{27}$$

where $\nabla = \partial_\mu \gamma_\mu$ and γ_μ are the Dirac matrices. Its three-dimensional transform

$$K_F(\vec{p}, t) = \frac{(i)(-i)}{2E}\{\theta(t)(\overset{+}{\mathbf{p}} + m)e^{-iEt} + \theta(-t)(\bar{\mathbf{p}} + m)e^{+iEt}\}, \tag{28}$$

where the sign $\pm$ on $\mathbf{p}$ indicates that the fourth component is $\pm E$. Multiplying by e^{ip_4t} and integrating with respect to t after introducing the factor $e^{-\epsilon t}$ to the first part and $e^{+\epsilon t}$ to the second, we get

$$\begin{aligned} K_F(p) &= \frac{i}{2E}\left\{\frac{\overset{+}{\mathbf{p}} + m}{p_4 - E + i\epsilon} - \frac{\bar{\mathbf{p}} + m}{p_4 + E - i\epsilon}\right\} \\ &= i\,\frac{\mathbf{p} + m}{p_4^2 - E^2} = \frac{i}{\mathbf{p} - m}, \end{aligned} \tag{29}$$

where the fourth component in $\mathbf{p}$ is p_4. It is important to note that the fourth component in $\overset{+}{\mathbf{p}}$ and $\bar{\mathbf{p}}$ are the positive and negative energies *on the mass shell* when the two parts are taken together, we obtain $\mathbf{p} + m$ with the fourth component *off the mass shell.* This is a remarkable feature which seems to have escaped attention since, from the inception of Feynman theory, no one was willing to tamper with the covariance of the propagator.

From the above derivation of four-dimensional transforms, it is clear that the propagator for both the bosons and fermions is obtained quite naturally *as a sum of two terms* corresponding to the positive and negative "arms." A Feynman diagram with n vertices and $n - 1$ fermion propagators breaks up into 2^{n-1} *patterns* if we distinguish between the positive and negative energy "arms." The matrix element of each pattern is characterized by the product of $(n - 1)$ energy denominators. Those corresponding to the positive energy are contribute denominators of the form $p_4 - E$ and those corresponding to negative energy $p_4 + E$. We shall now show that once this division is made these energy denominators are composed of *energy* im-

[1] Throughout this paper we shall use the standard Feynman notation [6]. For convenience, we represent a four-vector by an ordinary letter, its scalar product with Dirac matrices γ by the corresponding boldface letter and a three-vector by an arrow above it. The fourth component is represented by a subscript 4.

The u's are the usual spinors and $\tilde{u} = \bar{u}\gamma_4$, where $\bar{u}$ is transposed complex conjugate and $\tilde{u}$ the relativistic adjoint.

balances of vertices and that the denominators of field theory are composed of the *same imbalances*, grouped in a different manner. To establish the correspondence we first explain the concept of the energy imbalance of a vertex.

3. Energy Imbalance of a Vertex

We shall first define energy imbalance of a vertex when particle are assumed to have only positive energies. We shall then envisage negative energies and show that there are two choices possible for such negative energy particles—propagation forward or backward in time, and these correspond to the retarded and Feynman propagators, respectively. This elucidates immediately the subtle connection between the "general substitution rule" and the Feynman formulation of it.

To fix ideas we consider a vertex representing an incident fermion with momentum $\vec{p}$, an incident boson with momentum $\vec{q}$ and an emergent fermion with momentum $\vec{p} + \vec{q}$. Let the energies "on the mass shell" corresponding to these particles are $E(p)$, $E(q)$, and $E(p + q)$, respectively.

We call $E(p) + E(q) - E(p + q)$ as the energy imbalance of the vertex, i.e., the quantity representing the deviation from conservation or the difference between the energies of the incident and emergent systems.

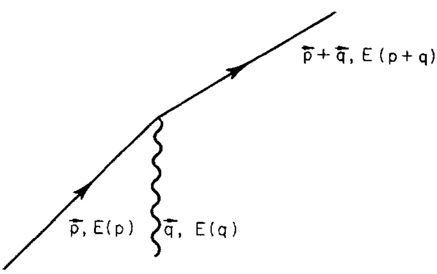


Fig. 1. A typical vertex.

Let us now take a vertex with incident fermion, boson, and antifermion with momenta $\vec{p}$, $\vec{q}$, $-(\vec{p} + \vec{q})$ and energies $E(p)$, $E(q)$, and $E(p + q)$, respectively.

The energy imbalance is obviously $E(p) + E(q) + E(p + q)$, since in this case all the three belong to the incident system in conventional field theory and there is no emergent system.

Now we study the consequences of applying the "general substitution rule," which we state as follows:

An incident particle (antiparticle) characterized by certain dynamical attributes and internal quantum numbers is equivalent to an emergent antiparticle (particle) with reversed dynamical attributes and internal quantum numbers.

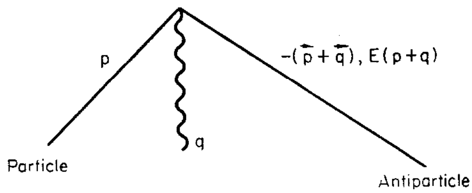


FIG. 2. Vertex with an incident antiparticle.

Application of this rule to the antiparticle implies that we can replace it by an emergent particle of momentum $+(\vec{p}+\vec{q})$ and energy $-E(p+q)$ which will travel forward in time, and the energy imbalance is

$$E(p)+E(q)+E(p+q).$$

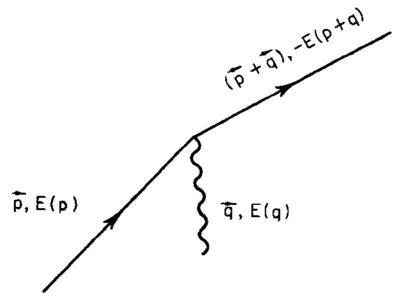


FIG. 3. The general substitution rule for antiparticle.

On the other hand, the Feynman substitution rule demands that the emergent negative energy particle travels backward in time but the energy imbalance is the same. This means that the imbalance can still be defined as the difference between the energies of the incident and emergent systems provided, incidence and emergence are defined by the directions *towards and away from the vertex*, respectively. The conservation of dynamical attributes and quantum numbers relate only to such *incident* and *emergent* systems and is not related to the particles travelling forward or backward to time.

The distinction between the two forms of the substitution rule has a consequence only when two vertices are joined by negative energy particles.

Thus for external lines the general and Feynman substitution rules have the same consequence, since in the computation of matrix elements we do not consider the propagation after emergence or before incidence.

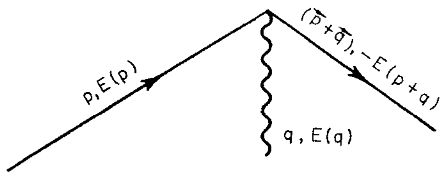


FIG. 4. Feynman substitution rule for antiparticle.

4. Composition of Imbalances

We can join two vertices if the emergent particle in one is identical with the incident particle in the other.

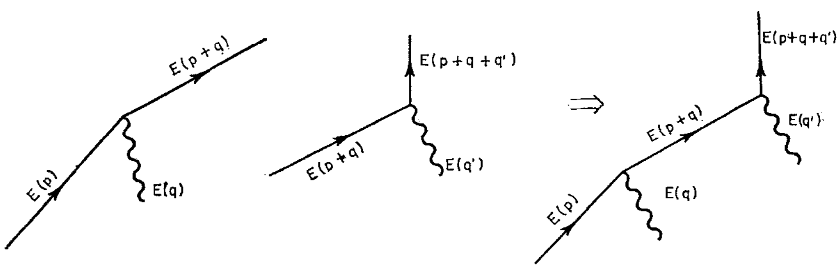


FIG. 5. Composition of two vertices.

Let the second vertex consist of an incident particle of momentum $\vec{p} + \vec{q}$ and energy $E(p + q)$ and emergent particles of momentum $\vec{q}\,'$ and $\vec{p} + \vec{q} - \vec{q}\,'$ and energies $E(q')$ and $E(p + q - q')$, respectively. The energy imbalance is then $E(p + q) + E(q') - E(p + q + q')$.

Denoting the imbalances by a_1 and b_1, respectively, the matrix element will contain the factor

$$\frac{1}{(a_1 + i\epsilon)(a_1 + b_1 + i\epsilon)}. \tag{30}$$

Consider now the case when we have at the first vertex an emergent particle of energy $-E(p + q)$ and at the second vertex, an incident particle of negative energy $-E(p + q)$. Since the negative energy particle travels back in time the second vertex occurs *earlier* in time.

The denominator corresponding to these two vertices is

$$(b_2 + i\epsilon)(b_2 + a_2 + i\epsilon), \tag{31}$$

where

$$\begin{aligned} a_2 &= E(p+q) + E(q) + E(p), \\ b_2 &= -E(p+q) + E(q) - E(p+q+q'). \end{aligned} \tag{32}$$

In Feynman theory the line connecting the two vertices is called a propagator, and no distinction is made between the two diagrams. The propagator is characterized by a quadratic denominator $p_4^2 - E^2$, where $p_4 = E(p) + E(q)$ and $E = E(p+q)$.

If the propagator is split into two parts with energy denominators $p_4 - E(p+q)$ and $p_4 + E(p+q)$, we identify them to be a_1 and a_2 corresponding to the first vertex in the two diagrams. If we now connect a third vertex to the second by the emergent fermion line, this is again a

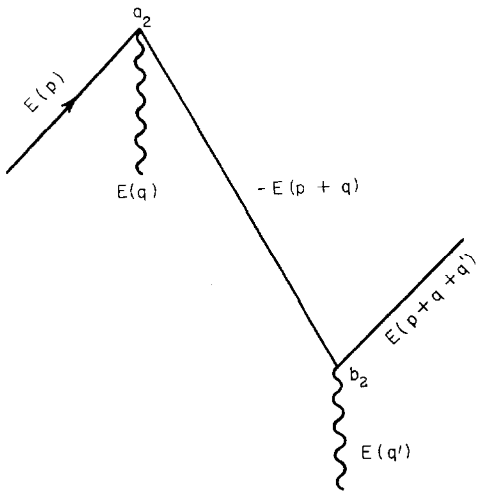


FIG. 6. Composition of two vertices.

propagator which has positive and negative energy arms. If we consider the positive arm for this second propagator the denominators corresponding to the two vertices for the two diagrams will be

$$(a_1 + i\epsilon)(a_1 + b_1 + i\epsilon)$$

and

$$(-1)(a_2 - i\epsilon)(a_2 + b_2 + i\epsilon),$$

respectively. The occurrence of $-i\epsilon$ in the denominator and the factor (-1) corresponds to the negative energy arm.

If this procedure of composition of vertices is adopted, it is quite clear that a single Feynman diagram of n vertices and $n-1$ fermion propagators (all boson lines external) is decomposed into 2^{n-1}, *patterns* and the energy imbalances characterize the vertices of a *pattern*. Corresponding to a particular pattern various time orderings of the vertices are possible consistent with the requirement that the relative time ordering of adjacent vertices should be the same for a pattern.

If the energy imbalances corresponding to a pattern of n vertices in Feynman sequence are

$$d(1), d(2),..., d(n),$$

then the process is real if

$$d(1) + d(2) + \cdots + d(n) = 0, \tag{33}$$

a condition representing the conservation of energy. The energy denominator can be written down as follows:

If this pattern is characterized by α negative energy arms corresponding to the ith, jth,..., propagators, then the energy denominator is of the form

$$\begin{aligned}(-1)^{\alpha}&[d(1) + i\epsilon][d(1) + d(2) + i\epsilon] \cdots [d(1) + d(2) + \cdots + d(i) - i\epsilon]\\ &[d(1) + d(2) + \cdots + d(i) + d(i+1) + i\epsilon]\\ &\cdots [d(1) + d(2) + \cdots + d(j) - i\epsilon] \cdots.\end{aligned} \tag{34}$$

The negative energy arms are represented by terms with $-i\epsilon$ and the factor (-1). This can be written in shorthand notation as

$$(-1)^{\alpha}[d(1), d(2),..., d^*(i),..., d^*(j),...]. \tag{35}$$

If a possible arrangement of the vertices in a temporal sequence is $l, m, n,...$, where the l'^s are just the above-mentioned d'^s then the energy denominator is

$$(l + i\epsilon)(l + m + i\epsilon)(l + m + n + i\epsilon)..., \tag{36}$$

which we write in shorthand notation as

$$[l, m, n,...]. \tag{37}$$

Our task is to show that the sum of field theoretic terms corresponding to a pattern, with all the factors in the denominators involving $+i\epsilon$ is equal to a term representing that pattern with some of the denominators involving $-i\epsilon$ and an additional factor (-1) corresponding to each such denominator. *Thus the reversal of $+i\epsilon$ and the occurrence of the factor (-1) turns out to be a simple but striking consequence of energy conservation. The summation yields certain denominators of the form*

$$[d(i+1) + d(i+2) + \cdots + d(n) + i\epsilon], \tag{38}$$

and from energy conservation we write this as

$$-[d(1) + d(2) + \cdots + d(i)] + i\epsilon \tag{39}$$

or

$$(-1)[d(1) + d(2) + \cdots + d(i) - i\epsilon], \tag{40}$$

which corresponds to the negative energy arm in a pattern.

A striking feature in the *numerator* of the fermion propagator has been pointed out earlier, as the occurrence of positive and negative energy "on the mass shell" in $\overset{\pm}{\mathbf{p}} + m$. This has immediately a very interesting consequence if we recognize that it is just the product of a spinor and its conjugate summed over spin states.

$$\overset{\pm}{\mathbf{p}} + m = \sum u_p \tilde{u}_p \tag{41}$$

and the factor $1/2E$ can be written as $1/\sqrt{2E}\,1/\sqrt{2E}$ so that to each spinor we can attribute the correct normalization factor. Since the propagator connects two vertices, we can ascribe one of the spinors to the emergent fermion and the other to the incident fermion at the vertices, respectively, and the energy denominator to the first vertex. Thus we have in the numerators spinors of particles with energies *on the mass shell*, the nonconservation in intermediate states being expressed through the denominators. Thus the numerators are characteristic of vertices only and since the vertices for all field theoretical terms corresponding to a pattern are *the same, the numerators corresponding to these field theoretical terms are the same.* Hence the task of summation just becomes that of adding reciprocals of products of denominators.

5. Application to a Process of Fifth Order

We shall demonstrate the equivalence for $n = 5$, when 120 field theoretical terms fall into 16 groups corresponding to 16 patterns.

The energy imbalances of the five vertices in Feynman sequence are

$$a_i\,,\, b_i\,,\, c_i\,,\, d_i\,,\, e_i\,,$$

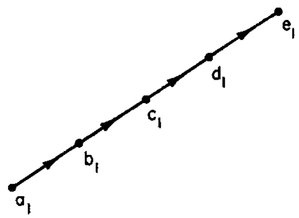
where the suffix i denotes the ith pattern and

$$a_i + b_i + c_i + d_i + e_i = 0. \tag{42}$$

We compute the terms representing all possible temporal orderings corresponding to a particular pattern, add them up and use:

Pattern 1

Only one field theoretical term corresponds to this pattern,

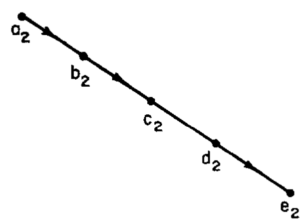


$$\frac{1}{[a_1\,,\, b_1\,,\, c_1\,,\, d_1]}\,,$$

which is the same as we write for the pattern.

Pattern 2

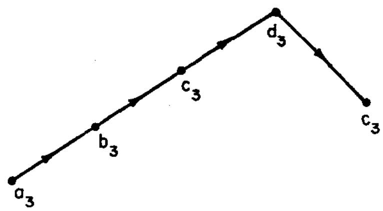
One field theoretical term corresponds to this pattern:



$$\frac{1}{[e_2\,,\, d_2\,,\, c_2\,,\, b_2]} = (-1)^4 \frac{1}{[a_2{}^*,\, b_2{}^*,\, c_2{}^*,\, d_2{}^*]}\,.$$

Pattern 3

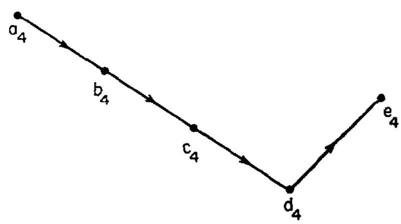
Four field theoretical terms correspond to this pattern:



$$\frac{1}{[a_3, b_3, c_3, e_3]} + \frac{1}{[a_3, b_3, e_3, c_3]} + \frac{1}{[a_3, e_3, b_3, c_3]} + \frac{1}{[a_3, b_3, c_3, d_3]}$$
$$= (-1)\frac{1}{[a_3, b_3, c_3, d_3{}^*]}.$$

Pattern 4

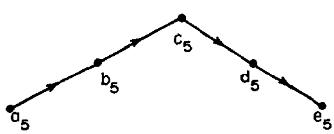
Four field theoretical terms correspond to this pattern:



$$\frac{1}{[d_4, e_4, c_4, b_4]} + \frac{1}{[d_4, c_4, e_4, b_4]} + \frac{1}{[d_4, c_4, b_4, e_4]} + \frac{1}{[d_4, c_4, b_4, a_4]}$$
$$= (-1)^3\frac{1}{[a_4{}^*, b_4{}^*, c_4{}^*, d_4]}.$$

Pattern 5

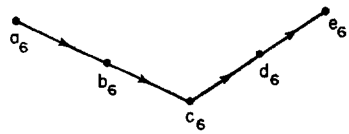
Six field theoretical terms correspond to this pattern:



$$\frac{1}{[a_5, b_5, e_5, d_5]} + \frac{1}{[a_5, e_5, b_5, d_5]} + \frac{1}{[a_5, e_5, d_5, b_5]} + \frac{1}{[e_5, a_5, b_5, d_5]}$$
$$+ \frac{1}{[e_5, a_5, d_5, b_5]} + \frac{1}{[e_5, d_5, a_5, b_5]} = (-1)^2\frac{1}{[a_5, b_5, c_5{}^*, d_5{}^*]}.$$

Pattern 6

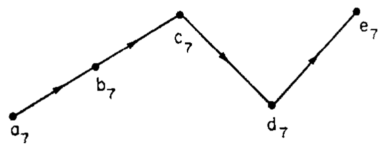
Six field theoretical terms correspond to this pattern:



$$\frac{1}{[c_6, d_6, b_6, e_6]} + \frac{1}{[c_6, d_6, e_6, b_6]} + \frac{1}{[c_6, b_6, d_6, e_6]} + \frac{1}{[c_6, d_6, b_6, a_6]}$$
$$+ \frac{1}{[c_6, b_6, d_6, a_6]} + \frac{1}{[a_6, b_6, c_6, d_6]} = (-1)^2 \frac{1}{[a_6{}^*, b_6{}^*, c_6, d_6]}.$$

Pattern 7

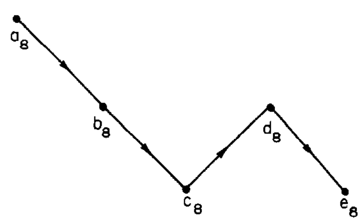
Nine field theoretical terms correspond to this pattern:



$$\frac{1}{[a_7, b_7, d_7, e_7]} + \frac{1}{[a_7, d_7, b_7, e_7]} + \frac{1}{[a_7, d_7, e_7, b_7]} + \frac{1}{[a_7, b_7, d_7, c_7]}$$
$$+ \frac{1}{[a_7, d_7, b_7, e_7]} + \frac{1}{[d_7, a_7, b_7, e_7]} + \frac{1}{[d_7, a_7, e_7, b_7]} + \frac{1}{[d_7, e_7, a_7, b_7]}$$
$$+ \frac{1}{[d_7, a_7, b_7, c_7]} = (-1) \frac{1}{[a_7, b_7, c_7{}^*, d_7]}.$$

Pattern 8

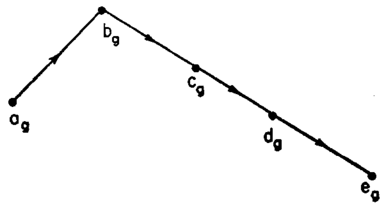
Nine field theoretical terms correspond to this pattern:



$$\frac{1}{[e_8, c_8, b_8, a_8]} + \frac{1}{[e_8, c_8, d_8, b_8]} + \frac{1}{[e_8, c_8, b_8, d_8]} + \frac{1}{[c_8, e_8, b_8, a_8]}$$
$$+ \frac{1}{[c_8, b_8, e_8, a_8]} + \frac{1}{[c_8, b_8, a_8, e_8]} + \frac{1}{[c_8, e\ , b_8, d_8]} + \frac{1}{[c_8, e_8, d_8, b_8]}$$
$$+ \frac{1}{[c_8, b_8, e_8, d_8]} = (-1)^3 \frac{1}{[a_8{}^*, b_8{}^*, c_8, d_8{}^*]}.$$

Pattern 9

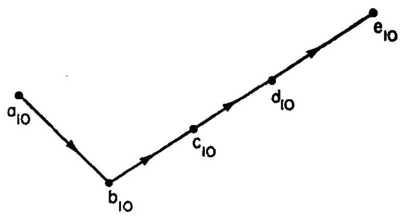
Four field theoretical terms correspond to this pattern:



$$\frac{1}{[a_9\,, e_9\,, d_9\,, c_9]} + \frac{1}{[e_9\,, a_9\,, d_9\,, c_9]} + \frac{1}{[e_9\,, d_9\,, a_9\,, c_9]} + \frac{1}{[e_9\,, d_9\,, c_9\,, a_9]}$$
$$= (-1)^3 \frac{1}{[a_9\,, b_9{}^*, c_9{}^*, d_9{}^*]}\,.$$

Pattern 10

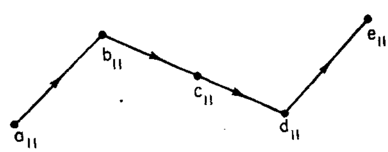
Four field theoretical terms correspond to this pattern:



$$\frac{1}{[b_{10}\,, a_{10}\,, c_{10}\,, d_{10}]} + \frac{1}{[b_{10}\,, c_{10}\,, a_{10}\,, d_{10}]} + \frac{1}{[b_{10}\,, c_{10}\,, d_{10}\,, a_{10}]}$$
$$+ \frac{1}{[b_{10}\,, c_{10}\,, d_{10}\,, e_{10}]} = (-1)\frac{1}{[a_{10}^*\,, b_{10}\,, c_{10}\,, d_{10}]}\,.$$

Pattern 11

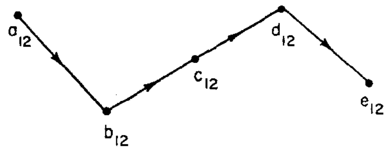
Eleven field theoretical terms correspond to this pattern:



$$\frac{1}{[a_{11}\,, d_{11}\,, c_{11}\,, e_{11}]} + \frac{1}{[a_{11}\,, d_{11}\,, e_{11}\,, c_{11}]} + \frac{1}{[a_{11}\,, d_{11}\,, c_{11}\,, b_{11}]}$$
$$+ \frac{1}{[d_{11}\,, a_{11}\,, c_{11}\,, e_{11}]} + \frac{1}{[d_{11}\,, c_{11}\,, a_{11}\,, e_{11}]} + \frac{1}{[d_{11}\,, a_{11}\,, e_{11}\,, c_{11}]}$$
$$+ \frac{1}{[d_{11}\,, c_{11}\,, e_{11}\,, a_{11}]} + \frac{1}{[d_{11}\,, e_{11}\,, a_{11}\,, c_{11}]} + \frac{1}{[d_{11}\,, e_{11}\,, c_{11}\,, a_{11}]}$$
$$+ \frac{1}{[d_{11}\,, a_{11}\,, c_{11}\,, b_{11}]} + \frac{1}{[d_{11}\,, c_{11}\,, a_{11}\,, b_{11}]} = (-1)^2 \frac{1}{[a_{11}^*\,, b_{11}\,, c_{11}^*\,, d_{11}]}\,.$$

Pattern 12

Eleven field theoretical terms correspond to this pattern:



$$\frac{1}{[e_{12}, b_{12}, c_{12}, a_{12}]} + \frac{1}{[e_{12}, b_{12}, a_{12}, c_{12}]} + \frac{1}{[e_{12}, b_{12}, c_{12}, d_{12}]}$$

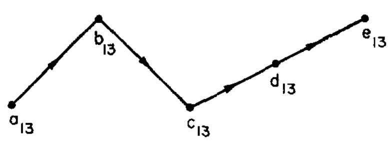
$$+ \frac{1}{[b_{12}, c_{12}, e_{12}, a_{12}]} + \frac{1}{[b_{12}, e_{12}, c_{12}, a_{12}]} + \frac{1}{[b_{12}, e_{12}, a_{12}, c_{12}]}$$

$$+ \frac{1}{[b_{12}, c_{12}, a_{12}, e_{12}]} + \frac{1}{[b_{12}, a_{12}, e_{12}, c_{12}]} + \frac{1}{[b_{12}, a_{12}, c_{12}, e_{12}]}$$

$$+ \frac{1}{[b_{12}, e_{12}, c_{12}, d_{12}]} + \frac{1}{[b_{12}, c_{12}, e_{12}, d_{12}]} = (-1)^2 \frac{1}{[a_{12}^*, b_{12}, c_{12}, d_{12}^*]}.$$

Pattern 13

Nine field theoretical terms correspond to this pattern:



$$\frac{1}{[a_{13}, c_{13}, d_{13}, e_{13}]} + \frac{1}{[a_{13}, c_{13}, b_{13}, d_{13}]} + \frac{1}{[a_{13}, c_{13}, d_{13}, b_{13}]}$$

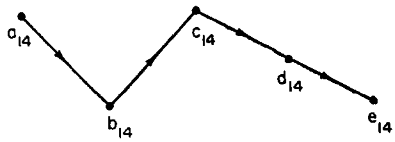
$$+ \frac{1}{[c_{13}, a_{13}, d_{13}, e_{13}]} + \frac{1}{[c_{13}, d_{13}, a_{13}, e_{13}]} + \frac{1}{[c_{13}, d_{13}, e_{13}, a_{13}]}$$

$$+ \frac{1}{[c_{13}, a_{13}, d_{13}, b_{13}]} + \frac{1}{[c_{13}, a_{13}, b_{13}, d_{13}]} + \frac{1}{[c_{13}, d_{13}, a_{13}, b_{13}]}$$

$$= (-(1 \frac{1}{[a_{13}, b_{13}^*, c_{13}, d_{13}]}.$$

Pattern 14

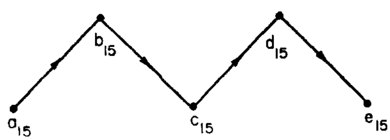
Nine field theoretical terms correspond to this pattern:



$$\frac{1}{[e_{14}, d_{14}, b_{14}, a_{14}]} + \frac{1}{[e_{14}, b_{14}, d_{14}, a_{14}]} + \frac{1}{[e_{14}, b_{14}, a_{14}, d_{14}]}$$
$$+ \frac{1}{[e_{14}, d_{14}, b_{14}, c_{14}]} + \frac{1}{[e_{14}, b_{14}, d_{14}, c_{14}]} + \frac{1}{[b_{14}, e_{14}, d_{14}, a_{14}]}$$
$$+ \frac{1}{[b_{14}, e_{14}, a_{14}, d_{14}]} + \frac{1}{[b_{14}, a_{14}, e_{14}, d_{14}]} + \frac{1}{[b_{14}, e_{14}, d_{14}, c_{14}]}$$
$$= (-1)^3 \frac{1}{[a_{14}^*, b_{14}, c_{14}^*, d_{14}^*]} .$$

Pattern 15

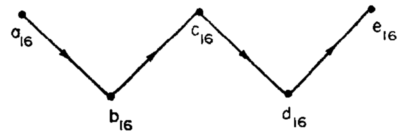
Sixteen field theoretical terms correspond to this pattern:



$$\frac{1}{[a_{15}, c_{15}, e_{15}, d_{15}]} + \frac{1}{[a_{15}, e_{15}, c_{15}, d_{15}]} + \frac{1}{[a_{15}, c_{15}, e_{15}, b_{15}]}$$
$$+ \frac{1}{[a_{15}, e_{15}, c_{15}, b_{15}]} + \frac{1}{[a_{15}, c_{15}, b_{15}, e_{15}]} + \frac{1}{[c_{15}, a_{15}, e_{15}, d_{15}]}$$
$$+ \frac{1}{[c_{15}, e_{15}, a_{15}, d_{15}]} + \frac{1}{[c_{15}, e_{15}, d_{15}, a_{15}]} + \frac{1}{[c_{15}, a_{15}, e_{15}, b_{15}]}$$
$$+ \frac{1}{[c_{15}, e_{15}, a_{15}, b_{15}]} + \frac{1}{[c_{15}, a_{15}, b_{15}, e_{15}]} + \frac{1}{[e_{15}, a_{15}, c_{15}, d_{15}]}$$
$$+ \frac{1}{[e_{15}, c_{15}, a_{15}, d_{15}]} + \frac{1}{[e_{15}, c_{15}, d_{15}, a_{15}]} + \frac{1}{[e_{15}, a_{15}, c_{15}, b_{15}]}$$
$$+ \frac{1}{[e_{15}, c_{15}, a_{15}, b_{15}]} = (-1)^2 \frac{1}{[a_{15}, b_{15}^*, c_{15}, d_{15}^*]} .$$

Pattern 16

Sixteen field theoretical terms correspond to this pattern:



$$
\begin{aligned}
&\frac{1}{[b_{16}, d_{16}, c_{16}, e_{16}]} + \frac{1}{[b_{16}, d_{16}, e_{16}, c_{16}]} + \frac{1}{[b_{16}, d_{16}, a_{16}, e_{16}]} \\
&+ \frac{1}{[b_{16}, d_{16}, e_{16}, a_{16}]} + \frac{1}{[b_{16}, a_{16}, d_{16}, e_{16}]} + \frac{1}{[b_{16}, d_{16}, a_{16}, c_{16}]} \\
&+ \frac{1}{[b_{16}, d_{16}, c_{16}, a_{16}]} + \frac{1}{[b_{16}, a_{16}, d_{16}, c_{16}]} + \frac{1}{[d_{16}, b_{16}, c_{16}, e_{16}]} \\
&+ \frac{1}{[d_{16}, b_{16}, e_{16}, c_{16}]} + \frac{1}{[d_{16}, e_{16}, b_{16}, c_{16}]} + \frac{1}{[d_{16}, b_{16}, a_{16}, e_{16}]} \\
&+ \frac{1}{[d_{16}, b_{16}, e_{16}, a_{16}]} + \frac{1}{[d_{16}, e_{16}, b_{16}, a_{16}]} + \frac{1}{[d_{16}, b_{16}, a_{16}, c_{16}]} \\
&+ \frac{1}{[d_{16}, b_{16}, c_{16}, a_{16}]} = (-1)^2 \frac{1}{[a_{16}^*, b_{16}, c_{16}^*, d_{16}]} .
\end{aligned}
$$

We hope that this systematic discussion draws attention to the fact that the decomposition of the propagators results in the use of spinors and wave functions of particles *on the mass shell*, the nonconservation being expressed through the energy denominators. This new view point we believe facilitates the understanding of concepts of charge conjugation, parity and time reversal *even in interaction* in the same manner as for free particles since in our perturbation expansions only "*on the mass shell*" terms occur. However, it is a curious fact that while the Feynman formalism is recognized to be equivalent to perturbative quantum field theory, transformations like charge conjugation, parity and time reversal are defined only for fields and have not yet been translated into the Feynman formalism. We now demonstrate that this procedure is not only possible but brings out in a very perspicuous manner the physical meaning of such operations for example charge-conjugation which look apparently formal in field theory.

6. Charge-Conjugation in Feynman Formalism

We shall first define charge conjugation in Feynman formalism and prove that it is identical with the conventional transformation in field theory.

According to Feynman rules in electrodynamics, following the usual notation,

(1) an incident electron of momentum p_1 is represented by

$$u(p_1)e^{-ip_1\cdot x},$$

where p_1 is the four momentum (energy positive).

(2) An emergent electron is to be represented by

$$\tilde{u}(p_2)e^{+ip_2\cdot x}$$

and the spinor $u(p)$ is the solution of

$$(\mathbf{p} - m)u(p) = 0$$

(3) is the charge of the electron is assumed to be e, (which is negative) and the photon is represented by A_μ then the perturbation is represented by $e\mathbf{A}$.

(4) An incident positron of four momentum p_1 (energy positive) is treated as a final electron of four momentum $-p_1$ (i.e., energy negative) and represented by

$$\tilde{u}(-p_1)e^{i(-p_1)\cdot x} = \tilde{v}(p_1)e^{-ip_1\cdot x},$$

where

$$u_{\pm 1/2}(\vec{p}, E) = v_{\mp 1/2}(-\vec{p}, -E),$$

the suffix referring to the spin.

(5) An emergent positron of momentum p_2 (energy positive) is treated as an incident electron of momentum $-p_2$ (energy negative) and represented by

$$u(-p_2)e^{-i(-p_2)\cdot x}.$$

The above procedure for the positron is expressed by the Feynman substitution rule

$$p_1 \to -p_2$$
$$p_2 \to -p_1 \,.$$

In this we keep the charge the same for the positive and negative electrons, the substitution rule safeguarding the properties of the positron. We shall call the world is which the above rules apply as the *Feynman world F* in which electrons are particles and positrons are antiparticles.

We now define the charge conjugated Feynman world F_c as that in which the positron is treated as a "particle" and the electron is an antiparticle and

(1) $u(p_1)e^{-ip_1\cdot x}$ represents an incident positron of four momentum (positive energy).

(2) $\tilde{u}(p_1)e^{+ip_2\cdot x}$ represents a final positron of four momentum p_2 (positive energy).

The positron in F_c has the same free-particle wave-function as electron in world F. However when it interacts with a photon in the world F_c, we assume that positron has charge $-e$ (which is positive since the charge e of the electron in F is negative) and its behavior is therefore different from the electron in F.

In F_c, an incident electron of four momentum p_1 is treated as a final state negative energy positron of four momentum $-p_1$ and is represented by

$$\tilde{u}(-p_1)e^{i(-p_1)\cdot x}.$$

An emergent electron of four momentum p_2 is treated as an initial negative energy positron of four momentum $-p_2$ and represented by

$$u(-p_2)e^{-i(-p_2)\cdot x}.$$

In F_c, the charge of the positive and negative positron is assumed to be $-e$ and perturbation due to interaction is represented by $(-e)$.

If we require that the electron scattering in F to be the same as an electron scattering in F, then we call the process invariant under charge conjugation. If in positron scattering in F, we change $\mathbf{A}$ to $-\mathbf{A}$, the matrix element is identical with that of electron scattering in F_c. Thus charge conjugation invariance can also be interpreted as requiring electron scattering in F to be identical with positron scattering in F itself when $\mathbf{A}$ is replaced by $-\mathbf{A}$.

Attention should be drawn to the fact that in going from electron to the positron in the same world we do not change the charge and this compels us to change $\mathbf{A}$ to $-\mathbf{A}$ to define charge conjugation invariance in a particular world. *This is identical with defining charge conjugation as going over from electron in F to an electron in F_c.* It is to be noted that the Feynman substitution principle helps us to represent either positive energy positrons in F in terms of negative energy electrons in F where electrons are particles, or positive energy electron in F_c in terms of negative energy positron in F_c where positrons are particles. Charge conjugation takes a positive energy electron in F_c to negative energy positron in F_c of a positron in F_c to a negative energy electron in F.

In any scattering process involving particles and antiparticles, we first use the substitution principle to redefine the scattering process in terms of *particles* only which may therefore have negative or positive energies and define the following six transformations only on the dynamical attributes.

(i) $p_x \to -p_x$, $s \to -s$

Reversal of x component of momentum and reversal of spin from z to $-z$ direction.

(ii) $p_x \to -p_y$, $s \to -s$

Reversal of p_y component of momentum and reversal of spin from z to $-z$ direction.

(iii) $p_z \to -p_z$

Reversal of the z component of momentum.

(iv) $E \to -E$

Reversal of energy.

(v) $E \to -E$, $p \to -p$ or reversal of the four dimensional momentum.

(vi) $m \to -m$ reversal of mass.

The matrix element is integral over space and time and intermediate momenta. Before these integrals are performed the integrand will contain both energy momentum, spin and space-time variables. *All transformations we have defined apply only to energy momentum and spin wherever they occur in the matrix element.*

The transformed matrix elements can also be obtained by the alternative prescription corresponding to the same transformation:

$$u(p) \to \gamma_x \gamma_5 u(p)$$

and

$$x \to -x,$$

i.e., we reverse only spatial $\times$ coordinate and operate $\gamma \times \gamma_5$ on the spinors wherever they occur in the matrix element. This is equivalent to the previously defined transformation of changing $p_x \to -p_x$ and $s_z \to -s_z$.

Similarly we identify operations corresponding to transformations on the dynamical attributes and obtain the following table:

Transformation in dynamical variables only	Transformation of space coordinates and the spinor only
$p_x \to -p_x$, $s_z \to -s_z$	$\gamma_x\gamma_5 \; -x$
$p_y \to -p_y$, $+s_z \to -s_z$	$\gamma_y\gamma_5 \; -y$
$p_z \to -p_z$	$\gamma_z\gamma_5 \; -z$
$E \to -E$	$\gamma_t\gamma_5 \; -t$
$p \to -p$	$\gamma_5 \; -x$
$m \to -m$	γ_5

A very interesting feature may be noted if we complex conjugate a spinor, we just change p_y to $-p_y$ in the spinor. Therefore this implies that the complex conjugate a spinor of spin s_z is obtained also by operating the spinor of spin $-s_z$ by $\gamma_2\gamma_5$.

We are now in a position to demonstrate the equivalence of the charge conjugation as defined above with the conventional operation in field theory. The equation for the interacting field is given as

$$(i\nabla - e\mathbf{A} + m)\psi = 0$$
$$A_\mu(x) = -J_\mu(x).$$

We define the charge conjugation C through

$$\psi_c = C\psi^* = \gamma_2\psi^*$$

by requiring ψ_c to satisfy the same equation as ψ if $\mathbf{A} \to -\mathbf{A}$. In the absence of interaction this is usually written in terms of free particle spinors

$$u^c_{\pm 1/2}(p) = cv_{\pm 1/2}(p).$$

By using the properties of $\gamma_2\gamma_5$ mentioned in the table

$$cv^*_{\pm 1/2}(p) = \gamma_2\gamma_2\gamma_5 v_{\mp 1/2}(p) = \gamma_5 u_{\pm 1/2}(-p)$$
$$= u_{\pm 1/2}(p)$$

and

$$[u_{\pm 1/2}(p)e^{-ip\cdot x}]^c = c[v_{\pm 1/2}(p)e^{+ip\cdot x}]^*$$
$$= u_{1/2}(p)e^{-ip\cdot x}.$$

If we change e to $-e$, we can interpret the above as the spinor of an incident positron in F_c. Thus the operation $u^c(p) = u(p)$ take us to the world F_c in which the charge is $-e$, while F refers to the world characterized by charge e.

We hope to apply these ideas to transformations like parity and time reversal and extend them even to cases when the "interaction" is switched on.

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